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Nonlinear Behavior of Rectangular Thin Plates Undergoing Free Vibration

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Abstract: *The present work is aimed at deriving the specific nonlinear frequency equations for the analysis of thin rectangular plates, based on the generally formulated equation, and using the new equations to predict the behavior of such plates under consideration. The shape profiles of the five plates under consideration were used to evaluate the individual plate types' stiffnesses due to bending and membrane actions. These were substituted into the general nonlinear frequency equation and evaluated to obtain the specific nonlinear equation for each plate type. The numerical results for the fundamental frequencies obtained for these plate types were compared with those in the literature. These values agree absolutely with those in literature with negligible percentage differences. It was observed that the nonlinear frequency increased as the ratio of out–of–plane displacement to the thickness of the plate (w/t) increased. The conclusion from this study is that the general nonlinear equation applies to these plate types for adequate prediction of the nonlinear frequency of thin rectangular plates. In addition, the approach is simpler and quicker for analysis, and has alleviate the dearth of numerical data for these plate types.*

Keywords: *Nonlinear Frequency, Specific Equations, Shape Profile, Free Vibration, Plate Stiffness*

INTRODUCTION

The behavior of structural steel or material is such that it is linear up to its proportional limit, and beyond its initial yield point it becomes nonlinear, that is, the stress is no longer proportional to the strain (Nash and Potter, 2011; Khurmi & Khurmi, 2013). This means that the young modulus 'E' of the material is no longer constant, but varies along the deformation line. At this inelastic stage, the E is smaller than that in the elastic range. The nonlinearity of a material can be expressed in terms of the material properties or the geometry of the material (Moy, 1981). The study of thin plates with respect to material nonlinearity only assumes that the deflection is very small comparatively, as such, the geometrical nonlinear effect is neglected. However, Ross (1987) stated that this assumption may not be true.

Several scholars have analyzed material nonlinearity problems using different approaches. The extremum principle of plasticity as reported by Prager and Hodge (1951), Hodge (1968), Johnson and Mellor (1973), and Martin (1975), is generally used to obtain upper and lower bounds for a true ultimate load of an elastoplastic plate in bending, because of the difficulties of the theoretical treatment of such plates. Lin and Ho (1968) and Lin (1968) applied the analogy concept to reduce the analysis of a plate with plastic strain to the analysis of identical plates with an additional set of lateral loads and edge moments. In terms of geometric nonlinearity, the effect of the geometrical change on the equilibrium of a plate must be considered as well as the stretching of the middle surface of the plate in analysis, the deflection of a plate is significant. Geometrical nonlinearity is accounted for by the addition of higher-order terms in the strain-displacement relations and /or by updating the geometry of the plate using a step-by-step approach. Most of the research works (Levy, 1942; Yamaki, 1959; Koiter, 1945 etc.) involving large deflection, are based on geometrical nonlinearity. It is observed that, without geometrical nonlinearity, the behavior of the plate cannot be investigated correctly, and without material nonlinearity, the ultimate strength cannot be accurately predicted (Lee, 1977). Structural nonlinearity is a structural system that results in having a stiffness matrix or load vector that is not constant (Klein, 2022). The formulated nonlinear frequency equation by Adah *et al*. (2021) incorporates both nonlinearities in its formulation of the expressions for dynamic analysis of thin rectangular isotropic plates under large deflection to predict the nonlinear frequency. This was done by the involvement of both strain-displacement relations and stress-strain relations in the formulation of the dynamic expression for thin rectangular isotropic plates under large deflection. Rao (2004) stated that based on Rayleigh's principle, the frequency of vibration of a conservative system vibrating about the equilibrium position, has a stationary value in the neighborhood of a natural mode; and that this value is a minimum value in the neighborhood of the fundamental natural mode. Thus, Rayleigh's method was fundamentally developed to evaluate only the fundamental natural frequency, of a given conservative system by using a single admissible function that satisfies all the geometric boundary conditions of the problem. Meirovitch (1986), gave the governing mathematical expression for Rayleigh's method as

$$\omega^2 = \frac{V_{max}}{T^*} \quad (1)$$

where V_{max} is the maximum potential energy, T^* is the reference kinetic energy, and ω is the circular frequency.

Other scholars who analyzed nonlinear free vibration problems are, Leissa (1973), Ventsel and Krauthermer (2001), etc. The use of Rayleigh's method, which often yields very good accuracy, is highly recommended if only the lowest natural frequency is required. This method is based on the principle of conservation of energy, while the Ritz method is based on the principle of minimum potential energy. Another method of analysis of nonlinear vibration problems is the Galerkin method. Reddy (1993) stated that the Galerkin method is a modified case of the Petrov-Galerkin method in which the approximation functions correspond to the weighted functions. In the Galerkin method, the variational integral is immaterial (Finlayson and Scriven, 1966). However, like in the Ritz method, the convergence of the Galerkin method depends very much on the selected weighted functions and their completeness. Leung and Mao (1995) used the method to work on the nonlinear vibration of beams and plates. Szilard (2004), stated that the Galerkin method can be applied successfully to such diverse types of problems as small and large deflection theories, linear and nonlinear vibration, and stability problems of plates and shells, provided that differential equations of the problem under investigation, have already been determined. The mathematical theory behind Galerkin's method is quite complicated, but its physical interpretation is relatively simple. The use of Galerkin's method is particularly recommended for the solution of differential equations with variable coefficients. Due to difficulties in the closed form or energy approach, the numerical solutions took the center stage to ease the difficulties of the former. Recently, Onodagu (2018), used Ritz Method to analyze the nonlinear dynamic problem of Thin Rectangular Plates. He did this by direct integration of Von Karman's large deflection governing equation based on polynomial displacement functions from the analytical energy approach. He applied the Ritz equation in his analysis and finally arrived at the nonlinear frequency of thin plates.

His approach was based on the long-standing von Karman large deflection equations of 1910, which are coupled, nonlinear partial differential equations with Airy's stress function. His mathematical model could predict a nonlinear frequency of minimal magnitude which seems to undermine the actual frequency of the plate beyond the yield point. However, Adah *et al.* (2021) circumvent the von Karman equation and formulated a general nonlinear frequency equation for free vibration analysis of thin isotropic rectangular as presented in Equation (2). This work aims to apply this newly formulated general equation to formulate specific equations for thin rectangular plates with different edge conditions and use them to predict the linear and nonlinear frequency.

MATERIALS AND METHODS

2.1 Linear/Nonlinear Resonating Frequency Equation

Adah *et al.* (2021), formulated a general nonlinear free vibration equation of a rectangular thin plate subjected to uniaxial loading along the x-axis as

$$\lambda_x = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\left(\frac{A}{t}\right)^2\right]} * \frac{1}{a^2}\sqrt{\frac{D}{\rho t}} \quad (2)$$

Where λ_x is the linear/nonlinear resonating frequency along the x-axis.

K_{bT} is the total bending stiffness given as

$$\mathrm{K_{bT}} = \left[\mathrm{k_{bx}} + \frac{2\mathrm{k_{bxy}}}{\partial^2} + \frac{\mathrm{k_{by}}}{\partial^4}\right] \quad (3)$$

K_{mT} is total membrane stiffness, given as

$$\mathrm{K_{mT}} = \left[\mathrm{k_{mx}} + \frac{2\mathrm{k_{mxy}}}{\partial^2} + \frac{\mathrm{k_{my}}}{\partial^4}\right] \quad (4)$$

A is the dynamic amplitude of vibration, given as

$$A = \frac{w}{h} \quad (5)$$

t is the thickness of the plate, k_λ is external dynamic load stiffness, a is the plate dimension along the x-axis, D is the flexural rigidity of the plate, and ρ is the plate density. W is the displacement of the plate, h is the shape profile of the plate, and the individual bending and membrane stiffnesses are given by

$$k_{bx} = \int_0^1 \left(\frac{\partial^2 h_x}{\partial R^2}\right)^2 \partial R * \int_0^1 {h_y}^2 \partial Q \quad (6a)$$

$$k_{bxy} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 \left(\frac{\partial h_y}{\partial Q}\right)^2 \partial Q \quad (6b)$$

$$k_{by} = \int_0^1 {h_x}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 \partial Q \quad (6c)$$

$$k_{mx} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^4 \partial R * \int_0^1 {h_y}^4 \partial Q \quad (6d)$$

$$k_{mxy} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 . {h_x}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 . {h_y}^2 \partial Q \quad (6e)$$

$$k_{my} = \int_0^1 {h_x}^4 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^4 \partial Q \quad (6f)$$

$$k_{Nx} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 {h_y}^2 \partial Q \quad (6g)$$

Subscripts b and m denote bending and membrane parts respectively.

At the point of maximum deflection, Equation (5) becomes

$$A = \frac{w}{h_{max}} \tag{7}$$

Substituting Equation (7) into Equation (2) yields

$$\lambda_{xmax} = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\left(\frac{1}{h_{max}}\right)^2\left(\frac{w}{t}\right)^2\right]} * \frac{1}{a^2}\sqrt{\frac{D}{\rho t}} \tag{8}$$

This reduces to

$$\lambda_{xmax} = \frac{f}{a^2}\sqrt{\frac{D}{\rho t}} \tag{9}$$

Where

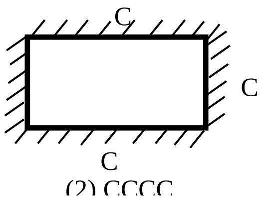
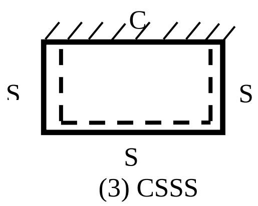
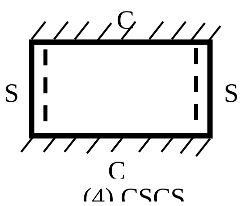
$$f = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\left(\frac{1}{h_{max}}\right)^2\left(\frac{w}{t}\right)^2\right]} \tag{10}$$

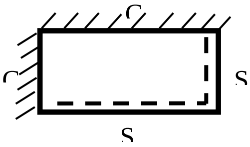
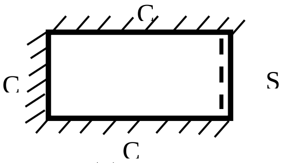
Is the coefficient of nonlinear resonating frequency. Equation (8) is the general nonlinear free vibration equation of a rectangular isotropic plate in terms of deflection w.

2.2 Formulation of Specific Nonlinear Equations for Five Plate Types

To formulate the specific nonlinear equations for these five plate types, the polynomial shape profiles of the plates under consideration are presented in Table 1.

Table-1 Plate Types, Shapes, and Shape Profile for Six Plate Types (Ibearugbulem *et al.*, 2014)

SN	PLATE BCs	SKETCHES (Plan View)	SHAPE PROFILE (h) W = Ah; (i.e h = R strip x Q strip)
1	CCCC = (C-C)R x (C-C)Q		$(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$
2	CSSS = (S-S)R x (C-S)Q		$(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$
3	CSCS = (S-S)R x (C-C)Q		$(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$

4	CCSS = $(C\text{-}S)_R \times (C\text{-}S)_Q$	 (5) CCSS	$(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$
5	CCCS = $(C\text{-}S)_R \times (C\text{-}C)_Q$	 (6) CCCS	$(1.5R^2 - 2.5R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$

2.3 Evaluation of Stiffness

Each of these shape profiles is evaluated using Equations 5 as follows: For CCCC plate, from Table-1, the polynomial deflected shape profile for the CCCC plate is given as;

$h = (R^2\text{-}2R^3\text{+}R^4)(Q^2\text{-}2Q^3\text{+}Q^4) = h_x * h_y$ (11)

Where

$h_x = (R^2\text{-}2R^3\text{+}R^4)$; $hy = (Q^2\text{-}2Q^3\text{+}Q^4)$ (12)

Differentiating Equations (11) and substituting into Equations (6a-g) and carrying out the evaluation of stiffness integrals we have the various stiffnesses of the CCCC plate as;

$$k_{bx} = \int_0^1 \left(\frac{\partial^2 h_x}{\partial R^2}\right)^2 \partial R * \int_0^1 {h_y}^2 \partial Q$$

$$k_{bx} = \int_0^1 (2 - 12R + 12R^2)^2 \partial R * \int_0^1 (Q^2 - 2Q^2 + Q^4)^2 \partial Q$$

$$k_{bx} = \int_0^1 (4 - 48R + 192R^2 - 288R^3 + 144R^4)\, \partial R * \int_0^1 (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)\, \partial Q \tag{13a}$$

$$k_{bxy} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 \left(\frac{\partial h_y}{\partial Q}\right)^2 \partial Q$$

$$k_{bxy} = \int_0^1 (2R - 6R^2 + 4R^3)^2 \partial R * \int_0^1 (2Q - 6Q^2 + 4Q^3)^2 \partial Q$$

$$k_{bxy} = \int_0^1 (4R^2 - 24R^3 + 52R^4 - 48R^5 + 16R^6)\, \partial R * \int_0^1 (4Q^2 - 24Q^3 + 52Q^4 - 48Q^5 + 16Q^6)\, \partial Q \tag{13b}$$

$$k_{by} = \int_0^1 {h_x}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 \partial Q$$

$$k_{by} = \int_0^1 (R^2 - 2R^2 + R^4)^2 \partial R * \int_0^1 (2 - 12Q + 12Q^2)^2 \partial Q$$

$$k_{by} = \int_0^1 (R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)\, \partial R * \int_0^1 (4 - 48Q + 192Q^2 - 288Q^3 + 144Q^4)\, \partial Q \tag{13c}$$

$$k_{mx} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^4 dRdQ = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^4 \partial R * \int_0^1 {h_y}^4 \partial Q$$

$$k_{mx} = \int_0^1 (2R - 6R^2 + 4R^3)^4 \partial R * \int_0^1 (Q^2 - 2Q^3 + Q^4)^4 \partial Q$$

$$k_{mx} = \int_0^1 (16R^4 - 192R^5 + 992R^6 - 2880R^7 + 5136R^8 - 5760R^9 + 3968R^{10} - 1536R^{11} + 256R^{12})\, \partial R * \int_0^1 (Q^8 - 8Q^9 + 28Q^{10} - 56Q^{11} + 70Q^{12} - 56Q^{13} + 28Q^{14} - 8Q^{15} + 16Q^{16})\, \partial Q \quad (13d)$$

$$k_{mxy} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 . {h_x}^2 \partial R * \int_0^1 \left(\frac{\partial h_y}{\partial Q}\right)^2 . {h_y}^2 \partial Q$$

$$k_{mxy} = \int_0^1 (2R - 6R^2 + 4R^3)^2 * (R - 2R^2 + R^4)^2 \partial R * \int_0^1 (2Q - 6Q^2 + 4Q^3)^2 * (Q - 2Q^2 + Q^4)^2 \partial Q$$

$$k_{mxy} = \int_0^1 (4R^6 - 40R^7 + 172R^8 - 416R^9 + 620R^{10} - 584R^{11} + 340R^{12} - 112R^{13} + 16R^{14})\, \partial R * \int_0^1 (4Q^6 - 40Q^7 + 172Q^8 - 416Q^9 + 620Q^{10} - 584Q^{11} + 340Q^{12} - 112Q^{13} + 16Q^{14})\, \partial Q \quad (13e)$$

$$k_{my} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial Q}\right)^4 dRdQ = \int_0^1 {h_x}^4 \partial R * \int_0^1 \left(\frac{\partial h_y}{\partial Q}\right)^4 \partial Q$$

$$k_{my} = \int_0^1 (R^2 - 2R^3 + R^4)^4 \partial R * \int_0^1 (2Q - 6Q^2 + 4Q^3)^4 \partial Q$$

$$k_{my} = \int_0^1 (R^8 - 8R^9 + 28R^{10} - 56R^{11} + 70R^{12} - 56R^{13} + 28R^{14} - 8R^{15} + 16R^{16})\, \partial R * \int_0^1 (16Q^4 - 192Q^5 + 992Q^6 - 2880Q^7 + 5136Q^8 - 5760Q^9 + 3968Q^{10} - 1536Q^{11} + 256Q^{12})^4 \partial Q \quad (13f)$$

$$k_\lambda = \int_0^1 \int_0^1 h^2\, dRdQ = \int_0^1 {h_x}^2 \partial R * \int_0^1 {h_y}^2 \partial Q$$

$$k_\lambda = \int_0^1 (R^2 - 2R^3 + R^4)^2 \partial R * \int_0^1 (Q^2 - 2Q^3 + Q^4)^2 \partial Q$$

$$k_\lambda = \int_0^1 (4R^4 - 4R^5 + 6R^6 - 4R^7 + R^8)\, \partial R * \int_0^1 (Q^4 - 4Q^5 + 6Q^6 - 4Q^7 + Q^8)\, \partial Q \quad (13g)$$

Integrating and substituting the values of R (= 0.5) and Q (= 0.5) at the point of maximum deflection, into Equations (12) yield the values of stiffnesses presented in Tables 4.5-4.11.
Similarly, the remaining four plate types are evaluated, and the individual stiffnesses are shown in Table-2.

Table-2 Summary of Polynomial-Based Plate Bending and Membrane Stiffnesses

BCs	k_{bx}	k_{bxy}	k_{by}	k_{mx}	k_{mxy}	k_{my}
CCCC	0.0012698413	0.0003628118	0.0012698413	0.0000000024	0.0000000005	0.0000000024
CSSS	0.0361904762	0.0416326531	0.0885714286	0.0000340978	0.0000045749	0.0000449254
CSCS	0.0076190476	0.0092517007	0.0393650794	0.0000016447	0.0000002610	0.0000019245
CCSS	0.0135714286	0.0073469388	0.0135714286	0.0000011786	0.0000001515	0.0000011786
CCCS	0.0028571429	0.0016326531	0.0060317460	0.0000000568	0.0000000086	0.0000000505

Table-3 Values of Plates Dynamic Load Stiffness for k_λ (Polynomial)

BCs	x-strip	y-strip	$k_{\lambda R}$ For x-strip	$k_{\lambda Q}$ For y-strip	$k_\lambda = k_{\lambda R} * k_{\lambda Q}$
CCCC	CC	CC	0.0015873016	0.0015873016	0.000002520
CSSS	SS	CS	0.0492063492	0.0075396825	0.000371000
CSCS	SS	CC	0.0492063492	0.0015873016	0.000078105
CCSS	CS	CS	0.0075396825	0.0075396825	0.000056847
CCCS	CS	CC	0.0075396825	0.0015873016	0.000011968

Substituting the individual stiffnesses from Table-2 and the dynamic external load stiffness values from Table-3 into Equation 1 for each of the plate types and evaluating yields the specific nonlinear equations for the coefficient of resonating frequency as presented in Table-4, and the resonating frequency equations for the specific plate types presented in Table-5. If we evaluate the shape profile at the point of maximum deflection, h_{max}, and substitute into Equation 9 together with the stiffnesses and evaluate, we will have the specific nonlinear resonating frequency equation for each plate type in terms of displacement w, as present in Table 6. Consider a square plate, that is, a plate with an aspect ratio of unity. The numerical results for the specific plates against w/t are present in Table for frequency parameter and Table 8 for nonlinear resonating frequency.

RESULTS AND DISCUSSION

Table-4 presents the coefficient of nonlinear formulated frequency equations for the five plate types under consideration, while the nonlinear formulated frequency equations for the five plate types are presented in Table-5 in terms of amplitude of displacement and Table-6, in terms of the maximum displacement itself. These are new equations for analysis of thin rectangular plates based on the considered plate types.

Table-4 Nonlinear Frequency, f, Coefficient Equation for the Various Plate Types (Polynomial)

Plate Type	$\lambda_x = \frac{f}{a^2}\sqrt{\frac{D}{\rho t}}; \quad \text{where } f = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\left(\frac{A}{t}\right)^2\right]}$
CCCC	$f = \frac{1}{ƨ^4}\left[\{504(ƨ^4+1) + 288ƨ^2\} + \{0.0014498048(ƨ^4+1) + 0.0005868258ƨ^2\}\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}$
CSSS	$f = \frac{1}{a^2}\left[(97.5483870968ƨ^4 + 224.4346349745ƨ^2 + 238.7368421053) + (0.1378616303ƨ^4 + 0.0369941212ƨ^2 + 0.1816389537)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$
CSCS	$f = \frac{1}{a^2}\left[(97.5483870968ƨ^4 + 236.9032258065ƨ^2 + 504) + (0.0315858688ƨ^4 + 0.0100233626ƨ^2 + 0.0369600013)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}$

CCSS	$$f = \frac{1}{a^2}\left[(238.7368421053(Ƨ^4+1) + 258.4819944598Ƨ^2) + (0.0310983350(Ƨ^4+1) + 0.0079936977Ƨ^2)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}$$
CCCS	$$f = \frac{1}{a^2}\left[(238.7368421053Ƨ^4 + 272.8421052632Ƨ^2 + 504) + (0.0071250276Ƨ^4 + 0.0021658503Ƨ^2 + 0.0063279075)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}$$

Table-5 Linear and Nonlinear Frequency λ_x Equations for the Various Plate Types (Polynomial)

Plate Type	$$\lambda_x = \frac{f}{a^2}\sqrt{\frac{D}{\rho t}};\quad \text{where } f = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\left(\frac{A}{t}\right)^2\right]}$$
CCCC	$$\lambda_x = \frac{1}{a^2Ƨ^2}\left[\{504(Ƨ^4+1) + 288Ƨ^2\} + \{0.0014498048(Ƨ^4+1) + 0.0005868258Ƨ^2\}\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$$
CSSS	$$\lambda_x = \frac{1}{a^2Ƨ^2}\left[(97.5483870968Ƨ^4 + 224.4346349745Ƨ^2 + 238.7368421053) + (0.1378616303Ƨ^4 + 0.0369941212Ƨ^2 + 0.1816389537)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$$
CSCS	$$\lambda_x = \frac{1}{a^2Ƨ^2}\left[(97.5483870968Ƨ^4 + 236.9032258065Ƨ^2 + 504) + (0.0315858688Ƨ^4 + 0.0100233626Ƨ^2 + 0.0369600013)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$$
CCSS	$$\lambda_x = \frac{1}{a^2Ƨ^2}\left[(238.7368421053(Ƨ^4+1) + 258.4819944598Ƨ^2) + (0.0310983350(Ƨ^4+1) + 0.0079936977Ƨ^2)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$$
CCCS	$$\lambda_x = \frac{1}{a^2Ƨ^2}\left[(238.7368421053Ƨ^4 + 272.8421052632Ƨ^2 + 504) + (0.0071250276Ƨ^4 + 0.0021658503Ƨ^2 + 0.0063279075)\left(\frac{A}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$$

Table-6 Nonlinear Frequency λ_{xmax} Equations for the Various Plate Types in terms of Deflection, w. (Polynomial)

Plate Type	$\lambda_{xMax} = \frac{f_w}{a^2}\sqrt{\frac{D}{\rho t}}$; where $f_w = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\frac{1}{(h_{max})^2}\left(\frac{w}{t}\right)^2\right]}$
CCCC	$\lambda_{xMax} = \frac{1}{a^2Ƨ^2}\left[\{504(Ƨ^4+1)+288Ƨ^2\} + \{95.0144088771(Ƨ^4+1)+38.4582131320Ƨ^2\}\left(\frac{w}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$
CSSS	$\lambda_{xMax} = \frac{1}{a^2Ƨ^2}\left[(97.5483870968Ƨ^4+224.4346349745Ƨ^2+238.7368421053) + (90.3489980269Ƨ^4+24.2444672683Ƨ^2 + 119.0389047079)\left(\frac{w}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$
CSCS	$\lambda_{xMax} = \frac{1}{a^2Ƨ^2}\left[(97.5483870968Ƨ^4+236.9032258065Ƨ^2+504) + (82.8004598680Ƨ^4+26.2756436762Ƨ^2 + 96.8884257826)\left(\frac{w}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$
CCSS	$\lambda_{xMax} = \frac{1}{a^2Ƨ^2}\left[\{238.7368421053(Ƨ^4+1)+258.4819944598Ƨ^2\} + \{127.3787800219(Ƨ^4+1)+32.7421855789Ƨ^2\}\left(\frac{w}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$
CCCS	$\lambda_{xMax} = \frac{1}{a^2Ƨ^2}\left[(238.7368421053Ƨ^4+272.8421052632Ƨ^2+504) + (116.7364530164Ƨ^4+35.4852920433Ƨ^2 + 103.6764367474)\left(\frac{w}{t}\right)^2\right]^{\frac{1}{2}}\sqrt{\frac{D}{\rho t}}$

Table-7 Numerical Values of Frequency Parameter $\frac{\rho\lambda_x{}^2a^4}{Et^2}$ for Various Plate Types.

$$\frac{\rho\lambda_x{}^2a^4}{Et^2} = \frac{1}{12(1-\upsilon^2)k_\lambda}\left[K_{bT} + \frac{3}{2}\left(\frac{A}{t}\right)^2 K_{mT}\right];\quad Ƨ = \frac{b}{a} = 1$$

w/t	CCCC	CSSS	CSCS	CCSS	CCCS
0	118.681	51.348	76.781	67.395	93.002
0.25	119.989	52.685	77.960	69.041	94.466
0.5	123.912	56.697	81.497	73.977	98.860
0.75	130.451	63.383	87.391	82.205	106.183
1	139.605	72.743	95.643	93.723	116.436
1.25	151.375	84.778	106.252	108.532	129.617
1.5	165.760	99.487	119.219	126.633	145.728
1.75	182.760	116.870	134.544	148.024	164.768
2	202.376	136.928	152.226	172.706	186.737
2.25	224.608	159.660	172.266	200.680	211.636
2.5	249.455	185.066	194.664	231.944	239.464
2.75	276.917	213.147	219.419	266.499	270.221
3	306.995	243.902	246.532	304.346	303.907

3.25	339.688	277.332	276.003	345.483	340.522
3.5	374.997	313.436	307.831	389.911	380.067
3.75	412.921	352.214	342.017	437.630	422.541
4	453.461	393.666	378.561	488.640	467.944
4.25	496.616	437.793	417.462	542.941	516.277
4.5	542.387	484.595	458.721	600.533	567.538
4.75	590.773	534.071	502.338	661.417	621.729
5	641.774	586.221	548.312	725.591	678.849

Table-8 Numerical Values of Nonlinear Resonating Frequency Coefficient, f, of Rectangular Plates for Various Plate Types.

$$\lambda_x = \frac{f}{a^2}\sqrt{\frac{D}{\rho t}}; \qquad Ƨ = \frac{b}{a} = 1$$

$$f = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\left(\frac{A}{t}\right)^2\right]} \quad \text{or} \quad f = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\frac{1}{(h_{max})^2}\left(\frac{w}{t}\right)^2\right]}$$

w/t	CCCC	CSSS	CSCS	CCSS	CCCS
0	36.000	23.680	28.956	27.129	31.868
0.25	36.198	23.986	29.177	27.458	32.118
0.5	36.785	24.882	29.832	28.422	32.857
0.75	37.743	26.309	30.892	29.961	34.052
1	39.045	28.184	32.317	31.991	35.658
1.25	40.657	30.426	34.063	34.426	37.622
1.5	42.545	32.960	36.081	37.186	39.892
1.75	44.674	35.724	38.330	40.205	42.418
2	47.010	38.668	40.771	43.428	45.157
2.25	49.525	41.755	43.372	46.813	48.074
2.5	52.192	44.955	46.106	50.327	51.137
2.75	54.990	48.245	48.950	53.946	54.321
3	57.900	51.608	51.886	57.649	57.608
3.25	60.905	55.031	54.899	61.422	60.980
3.5	63.992	58.504	57.979	65.252	64.423
3.75	67.150	62.018	61.113	69.130	67.928
4	70.369	65.566	64.295	73.048	71.484
4.25	73.641	69.143	67.518	76.999	75.085
4.5	76.960	72.745	70.776	80.980	78.724
4.75	80.320	76.368	74.064	84.986	82.397
5	83.715	80.010	77.379	89.014	86.099

The equations are unique and simple to apply. What will change is flexural rigidity due to the young modulus of the materials and the Poisson ratio. The numerical values for the frequency parameter and nonlinear frequency for the five plates as presented in Tables 7 and 8 respectively indicate that the plate sustains additional strength beyond the initial yield point. This is in line with the behavior of ductile plates unlike columns, plates do not fail at the initial yields. It is this property that makes the plate very useful in aerospace and shipbuilding industries since this property results in lighter weight but higher strength. Before the plate starts deflecting or bending, that is at the point w =0, the plate reaches its fundamental natural frequency. At this point linear analysis is applicable and the second term of these equations becomes equal to zero.

Beyond this fundamental frequency value, the plate increases in frequency magnitude as the deflection increases. To validate the results of this work, a comparison was carried out between the fundamental frequency predicted by these equations for the specific plate types with the results obtained by Ibearugbulem *et al.,* (2014) and Leissa and Quta (2011) as presented in Table-9. From the analysis, it is shown that the percentage difference among them is negligible, hence indicating that the formulated equations are adequate for predicting the fundamental frequency of thin rectangular plates. In addition, the result indicated that a plate clamped at all sides (CCCC) has a higher fundamental frequency than a plate mixed with simple support. This implies that a CCCC plate is safer than others since they will not easily reach the resonance point. Also, a comparison of the nonlinear frequency of the CCCC plate obtained from this equation using a polynomial shape profile with the values obtained using a trigonometric shape profile as presented in Table 10 indicated a close match between the two shape profiles. From the results, it is shown that the polynomial shape profile is lower bound to the results of the trigonometric shape profile with the highest percentage difference of 6.4048, which is still small and acceptable within statistical limits. There is a dearth of nonlinear frequency data available for these plate types except for the SSSS plate obtained by Levy in 1945. Since the equations predict adequately the fundamental frequency of the plates considered and the predicted result agrees with the practical behavior of a thin rectangular plate (that is, an increase in frequency beyond the initial yield point), then, it is logical to conclude that the formulated equations are adequate to analysis linear and nonlinear frequency of thin plates. And that the generally formulated free vibration equation for nonlinear frequency analysis applies to these plate types and not just to the SSSS plate only.

Table-9 Comparison of Resonating Natural (Linear) Frequency with those in Literature for Aspect Ratio of Unity.

Plate Type	Present work	Ibearugbulem et. al, 2014	Leissa & Quta, 2011	%Difference	%Difference
CCCC	36.000	35.967	35.99	0.091751	0.027785
CSSS	23.680	23.680	23.646	0	0.143788
CSCS	28.936	28.959	28.951	-0.07942	-0.05181
CCSS	27.129	27.129	27.06	0	0.254989
CCCS	31.868	31.868	31.83	0	0.119384

Table-10 Nonlinear Resonating Frequency Parameter values and Nonlinear Resonating Frequency, for Aspect Ratio, Ƨ = 1 for CCCC plate.

w/t	$\frac{\rho\lambda^2 a^4}{Et^2}$		% Difference 100(F₁-F₂)/F₁	$\lambda_{Max} = \frac{f}{a^2} * \sqrt{\frac{D}{\rho t}}$ $f = \sqrt{\left[\frac{K_{bT}}{k_\lambda} + \frac{3}{2}\frac{K_{mT}}{k_\lambda}\frac{1}{(h_{max})^2}\left(\frac{w}{t}\right)^2\right]}$		%Difference 100(f₁-f₂)/f₁
	Poly. F_1	Trig. F_2		Poly. F_1	Trig. F_2	
0	118.6810	126.8660	-6.8966	36.0000	37.2210	-3.3917
0.25	119.9890	128.3760	-6.9898	36.1980	37.4410	-3.4339
0.5	123.9120	132.9050	-7.2576	36.7850	38.0960	-3.5640
0.75	130.4510	140.4550	-7.6688	37.7430	39.1630	-3.7623
1	139.6050	151.0250	-8.1802	39.0450	40.6100	-4.0082
1.25	151.3750	164.6140	-8.7458	40.6570	42.3980	-4.2822
1.5	165.7600	181.2230	-9.3285	42.5450	44.4860	-4.5622
1.75	182.7600	200.8530	-9.8999	44.6740	46.8330	-4.8328
2	202.3760	223.5020	-10.4390	47.0100	49.4030	-5.0904
2.25	224.6080	249.1710	-10.9359	49.5250	52.1630	-5.3266

2.5	249.4550	277.8590	-11.3864	52.1920	55.0840	-5.5411
2.75	276.9170	309.5680	-11.7909	54.9900	58.1420	-5.7320
3	306.9950	344.2970	-12.1507	57.9000	61.3170	-5.9016
3.25	339.6880	382.0450	-12.4694	60.9050	64.5910	-6.0520
3.5	374.9970	422.8130	-12.7510	63.9920	67.9490	-6.1836
3.75	412.9210	466.6020	-13.0003	67.1500	71.3810	-6.3008
4	453.4610	513.4100	-13.2203	70.3690	74.8760	-6.4048

CONTRIBUTION TO KNOWLEDGE

This work has applied the general formulated linear/nonlinear resonating frequency equation to developed new specific linear/nonlinear resonating frequency equations for five thin rectangular plate types using both polynomial shape functions. Based on the adequacy of the numerical results, the present work has proven that the new general formulated linear/nonlinear resonating frequency equation is applicable to thin rectangular plate types other than a plate simply supported all-round (SSSS). This will relieve in no small measure the cumbersome effort put by plate analysts and designers in solving plate problems.

CONCLUSION

Specific equations for nonlinear free vibration analysis for five plate types have been formulated. The numerical results predicted by these equations have shown to be adequate when compared with the available data in the literature. The results also, agree with the behavior of thin plates, which is another indicator that the equations are adequate. Based on these results, it means the general nonlinear frequency equation (Equations 1 and 7) is applicable for the analysis of thin rectangular plates other than SSSS plate type and these equations are a simpler, quicker, and improved alternative to the long-standing von Karman equations for the analysis of plates under large deflection formulated in 1910.

CONFLICT OF INTEREST

There is no conflict of interest for this research work.

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